

A STUDY OF THE CURRENT REVOLUTION IN
MATHEMATICAL INSTRUCTION AND PHILOSOPHICAL
ASPECTS IN RELATION TO TEACHING PLANE GEOMETRY

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"We challenge the United States to competition in education," declared the Department of Education of the Union of Soviet Socialist Republics at an international conference on the teaching of mathematics, July 1956, in Geneva.¹ Why was this challenge issued with such confidence? Perhaps, it was the fact that Russia had realized that mathematical science is the cornerstone to technological advancement in a world which is becoming more science-orientated. While Russia was planning to develop the mathematical potential of every high school, the United States, at this time, had no such long-range plan. This statement served as a rude stimulus to awaken all nations, particularly the United States, to the reality that they have been lacking in first rate programs of mathematical instruction in their educational institutions. The high school curriculum of mathematical instruction in the United States is no longer suited to the current needs and conditions of life; hence, educators, as well as mathematicians, have re-examined and re-evaluated the mathematics that is being taught today, both philosophically and pedagogically, in efforts to improve the curriculum of mathematical instruction.

Russia is not entirely the cause of the present concern with mathematics, however, but rather the breathtaking movement of the United States into a new technological era has

¹National Council of Teachers of Mathematics, The Revolution in School Mathematics, p.15.

created new needs for mathematics.² (In fact, curriculum revision started as early as 1952 at the University of Illinois long before the threat of Russia.) Mathematics is becoming increasingly recognized as the true foundation of technology, and the society of the United States is ever more dependent upon mathematics.

This long term revision was obscured first by the depression of the 1930's, and secondly by the dislocation of World War II. The realization of the true situation occurred with suddenness in the 1950's, long after efforts should have been initiated.³ This technological revolution, now in progress and recognized by all, requires that new approaches to mathematics be taught in the schools so as to yield better and more productive scientists, mathematicians, teachers, and others engaged in occupations involving a knowledge of mathematics. With such new approaches, it is hoped that the potential of every individual in mathematics will be developed to its utmost capacity. This demand for quality as well as quantity is actually part of a larger demand for highly trained personnel in all fields of endeavor.⁴ This demand is spurred further by a human drive to improve civilization and society.

The current revolution in mathematics has also been

²Report of the Commission on Mathematics, Program for College Preparatory Mathematics, p. 9.

³National Council of Teachers of Mathematics, op. cit., p. 10.

⁴Ibid., p. 10.

greatly influenced by the automation revolution. Automation has made possible the development of machines of enormous size, complexity, and costs⁵ Interesting problems involved in the construction and operation of these machines have evolved as a result--problems which demand solutions. As a result, a terrific need has been created for people with the ability to do abstract thinking; even computers cannot solve problems without careful programming by man. It is the general consensus among mathematics educators that traditional mathematics, as taught in many educational institutions, simply does not foster creative thinking. One very important reason why there has been little emphasis placed upon abstract thinking in traditional mathematics courses is because the needs of mathematics in the past did not demand this. For example, consider the needs of mathematics in the middle of the nineteenth century. The majority of problems involving the use of mathematics were simply those which utilized arithmetic computations such as keeping simple accounts and working problems of measurements.⁶ The public school curriculum in traditional mathematics, stressing the utility of mathematics, has included a treatment of all of these topics and have stressed rote memorization in addition to extensive drill in the form of endless lists of mechanical exercises. As newer needs for mathematics have developed, tradi-

⁵Ibid., p. 3.

⁶Ibid., p. 5

tional mathematics has incorporated more subject matter with a recurring emphasis on drill and memorization. Such an approach to mathematics does not meet the needs of advancing technology today. There must be a shift in the emphasis of subject matter taught in traditional mathematics in addition to a shift in the pedagogical and philosophical approaches to meet the needs of society.⁷

Learning theories formulated by leading psychologists have had a tremendous influence upon educational practices, including mathematical instruction.⁸ Throughout the first half of the twentieth century, Thorndike's psychological explanation of learning dominated the scene. Thorndike advocated that learning consists of a response, given a stimulus. Connected with this theory are extensive drill and laws of readiness; both of these are characteristic of traditional mathematics. Pavlov, another famous theorist, believed learning could be explained through the phenomenon of conditioned response. That is, a given response could be expected through conditioning. This particular learning theory is employed extensively in areas involving verbalized skills, such as foreign languages.⁹

For about three decades Gestalt field psychology has intrigued educators; it is increasingly replacing Thorndike's

⁷National Council of Teachers of Mathematics, The Revolution in School Mathematics, p. 5.

⁸National Council of Teachers of Mathematics, The Growth of Mathematical Ideas: Grades K-12, p.408.

⁹Alice Crowe, Educational Psychology, p. 146.

influence upon educational practices and has produced the most serious challenge to the stimulus-response theory of learning.¹⁰ According to the Gestalt point of view, as many situations as possible should be viewed as functional wholes rather than separated into parts.¹¹ Gestalt psychologists believe that insight comes with sudden clarity and that once reached needs little repetition. Understanding the structure means understanding the problem.¹² The Gestalt theory of learning is reflected throughout these revised programs of mathematical instruction.

The tremendous advance made in the area of pure mathematical research has also strongly influenced the revision of mathematical instruction. New ideas and concepts have been formulated which lead to a better understanding of the structure of mathematics. Vector analysis, taught in conjunction with plane geometry, is one example. Theorems which have been accepted in the past have become somewhat obsolete since new and better ones have been discovered; inconsistencies have also been found in certain axiomatic structures. Consider Euclidean geometry, for example, which is the basis for traditional plane geometry. Recent developments in geometrical thinking have disclosed grave defects in the logical structure of Euclid, and thus attention has been given to the need of modification in

¹⁰National Council of Teachers of Mathematics, op. cit., p.408.

¹¹Alice Crowe, op. cit., p. 146.

¹²National Council of Teachers of Mathematics, op. cit., p.408.

the traditional approach to high school geometry.¹³ The geometry of Euclid has been, and still is, a great contribution to mathematics, but his major handicap was that he lacked an adequate algebra necessary to create a rigorous, consistent set of postulates. "Euclid is the only man to whom there ever came or ever will come the glory of having successfully incorporated in his ratings all the essential parts of the accumulated (mathematical) knowledge of his time."¹⁴

Euclid tried to be as rigorous as possible and tried to avoid the use of intuitive geometry. He was interested in the systemization of geometric facts, and not in their discovery; his geometry was also written for the scholars and philosophers of his day and not for the schoolboy.

Euclid recognized the necessity of starting with appropriate definitions and assumptions. He failed to recognize the necessity of primitive notion (undefined terms) and went to unnecessary and inadvisable lengths to define every term.¹⁵ A point, for instance, is defined to be that which has no part, and a line is that which has breadthless length. Euclid defines lines and points in relation to other terms which he fails to define or ever use. In order to avoid circumlocution, certain terms should be undefined; Euclid's definitions play

¹³Report of the Commission on Mathematics, op. cit., p. 3.

¹⁴D. E. Smith, History of Mathematics, p. 102.

¹⁵Leonard Blumenthal, A Modern View of Geometry, p. 3.

no role whatsoever in the logical development of geometry.¹⁶⁷

Many of Euclid's postulates are not stated precisely, and if a geometry is to be presented on a strictly logical basis, then it is necessary to say what is meant or else be misinterpreted. Throughout his system of postulates, he uses words which describe physical activities and interprets them differently. The wording of his first postulate which states "it is possible to draw a straight line from any point to any point" makes the meaning rather unclear. The word "draw" actually refers to a physical activity. Euclid interprets "draw" to mean "there exists" rather than the connotation of its being a physical activity. The meaning of his postulates are often misinterpreted as a result.

The major defect of Euclid is his failure to recognize the necessity of making formal assumptions concerning "betweenness"-an omission first recognized by Gauss. For any logical formulation of geometry, certain assumptions about the order of points should be postulated.¹⁷ Euclid also failed to formulate explicitly the axiomatic basis upon which his congruence theorems rest.¹⁸ Although he argues in support of the superimposition of line segments, the less obvious manner of justifying this superimposition is neglected.¹⁹ This defect could have been remedied

¹⁶School Mathematics Study Group, Studies in Mathematics, p. 1.16.

¹⁷Report of the Commission on Mathematics, op. cit., p. 23-24.

¹⁸Ibid.

¹⁹Ibid.

by assuming the congruence theorems outright.²⁰

Euclid's geometry has also served as a basis for non-Euclidean geometries which substitute various postulates for the parallel postulate of Euclid: "Through any point not incident with a given line there passes one and only one line that does not intersect the given line." Non-Euclidean geometries replace this postulate. The fact that this parallel postulate can be replaced by another and still result in a consistent set of postulates "was not one of Euclid's mistakes, but one of his crowning achievements of mathematics."²¹ Thus the works of Lobachevski, founder of hyperbolic geometry; Riemann, founder of elliptic geometry; and many others, could rely heavily upon the elements of Euclidean geometry.

Attempting to remedy the defects in Euclid's system, many mathematicians have improved his postulate system. Such improvements have had a profound influence upon the revolution in mathematical instruction, as will be discussed in this paper.

Newspapers, magazines, and authors often attach the term "modern mathematics" or "new math" to the revision in mathematics attributed to the current revolution. Modern mathematics, as the term is popularly used, does not mean necessarily that the concepts or notions of mathematics are new, but that the philosophical and pedagogical approach is different as compared with that

²⁰ Ibid.

²¹ School Mathematics Study Group, op. cit., p. 1.14.

of the traditional courses. This current reform in the mathematics curriculum actually means a tremendous development quantitatively in the content of mathematics, a change in approach to the familiar content, and changes in the methods of teaching and preparation of teachers.²² Some new material is introduced, and much of the traditional content is rearranged or extended; but the subject matter is basically the same as that taught in traditional courses.²³ The pedagogical differences between the revised program of mathematical instruction and the traditional program are the most significant and, by far, the most important since they involve methods of approach to effective learning. These revised programs seek to develop many promising pedagogical techniques and approaches necessary to accomplish their individual objectives.

Two important points to consider are the value of these revised programs and common philosophical objectives. One of the basic objectives of these revised programs is to place in proper balance the student's memorization of methods, rules, and facts; and on the other side of the scale, his reasoning power to do abstract thinking rather than the mechanical aspects as stressed in traditional programs. These revised pro-

²²National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs, p. 58.

²³W. Eugene Fergenson, "Current Reforms in the Mathematics Curricula--a Passing Phrase or Progress?", Mathematics Teacher, LVII:143, March, 1964.

grams advocate integration of drill and theory, unlike traditional mathematics programs, which through seemingly endless lists of numerical exercises, treat drill and theory as separate entities.²⁴ Although modern mathematics curriculums see the need for computation, greater emphasis is placed upon student discovery and the appreciation of the structures of mathematics. Thus, these programs, putting less emphasis upon computations, seek to make mathematics more stimulating and more enjoyable to the individual. Mathematics, in these modern programs, is not taught for its utilitarian value only, as emphasized in traditional programs, but for the intellectual stimulation and excitement gained through understanding. The attitudes of these programs are:

"You must interest adolescents in ideas. It is of little value to try to obtain student interest by promises of utility in adult life. Most high school students are not genuinely stirred by such sales campaigns. The goal of educational utility is too remote to make much difference to a ninth grader. He wants to know how mathematics fits into his world, and happily, his world is full of fancy and abstractions, because it gives him access to a kind of intellectual thrill that is enticing and fancy."²⁵

One of the characteristics of contemporary mathematics is its renewed, increased, and conscious emphasis that mathematics is concerned with abstract patterns of thought.²⁶

Precisely what the new mathematics demand is that new roots, as well as efficient ones, be found in the foundation of

²⁴Edwin Moise, "The MSG Geometry Program," The Mathematics Teacher, LIII:440, Oct., 1960.

²⁵National Council of Teachers of Mathematics, The Revolution

²⁶Report of the Commission on Mathematics, p. 3.

the subject as layed in secondary schools to the newer approaches without laboriously traversing all of the older content.²⁷ In the process much of the traditional materials which are obsolete will be dropped. While still of value, these materials are of lesser value than the objective of obtaining a true understanding and appreciation of the methods, content, and beauty of mathematics.

Recognizing the need for the improvement of school mathematics, several national foundations have contributed large sums of money for experimentation purposes in the area of curriculum improvement. The federal government, for example, through the National Defense Foundation, has contributed much money for experimental work. From such assistance, committees were formed to study mathematics curriculums and make recommendations for improvements; and study groups, whose major purpose has been to revise the mathematics curriculum at both the elementary and the secondary levels, were initiated. Although there are many similarities of ideas among these groups, some of which were pointed out previously, there also exist some significant differences with regard to their philosophical objectives, course content, and pedagogy. To better examine these basic differences, particularly in relation to geometry, consider the three major study groups - School Mathematics Study Group, University of Illinois Committee on School Mathematics, and Ball State Experimental Program - and

²⁷National Council of Teachers of Mathematics, p. 70.

a brief survey of their origins and general objectives. In order to further acquaint the reader with these possibly unfamiliar programs, the report of the Commission on Mathematics, "Programs for College Preparatory Mathematics," which consists of recommendations for curriculum revision, will also be reviewed.

Commission on Mathematics
of the College Entrance Examination Board

This commission was set up to broadly consider the college preparatory mathematics curriculum in the secondary schools and to make recommendations for its modernization, modification, and improvement. The main objective of this commission has been to produce recommendations for a college preparatory curriculum which is oriented to the needs of mathematics, natural science, social science, business, and industry in the second half of the twentieth century.²⁸ This report was formulated between the years of 1955 and 1959 with concentration upon grades nine through twelve. The Commission has not set up any particular curriculum or written a set of texts; however, many recommendations of the Commission have had a profound influence upon study groups - especially the School Mathematics Study Group - in the revision of the mathematics curriculum. Some of the general proposals of the Commission are as follows:

1. strong preparation in both skill and concept;

²⁸Scott, Foreman and Co., Studies in Mathematics Education, p. 26.

2. understanding of the nature and roll of deductive reasoning;
3. appreciation of mathematical structures;
4. responsibility of the schools to those students entering college;
5. concentration on long range ideas, not on topics of immediate utility which do not contribute to the basic mathematical knowledge of the students;
6. greater emphasis upon general principles, ideas, and techniques that have a wide application and educational value;
7. judicious use of unifying ideas;
8. elimination of present topics such as deductive solid geometry as a course in itself, including it in other places;²⁹
9. presentation of old topics in a new way;
10. introduction of new topics - sets, probability, statistics, abstract algebra, logic, analytic geometry, elementary calculus;
11. incorporation of coordinate geometry with plane geometry, and the essentials of solid geometry with space perception.³⁰

University of Illinois

Committee on School Mathematics

Since 1952, the University of Illinois Committee on School Mathematics has been working on materials of mathematical instruction, the development of teaching methods, and the training of teachers for a new curriculum in mathematics for the se-

²⁹Ibid., p. 27.

³⁰Report of the Commission on Mathematics, op. cit. p. 26.

condary schools. This study group was the first study group formed to revise the mathematics curriculum in the schools; it is also the first to employ the Gestalt theory of learning. UICSM, as it is often called, has developed student texts and teacher guides for grades seven through twelve. This program is under the direction of Dr. Max Beberman and the group consists of a joint committee of the College of Education, the College of Engineering, and the College of Liberal Arts and Sciences.³¹ The project is supported largely by the Carnegie Corporation from funds amounting to over one-half million dollars.

The textbooks emphasize consistency, precision of language, structures of mathematics, and understanding of basic principles through pupil study. Discovery of generalizations by the student is the basic technique used throughout the program. Some of the primary convictions of UICSM are as follows: ³²

1. a consistent exposition of high school mathematics;
2. great interest of high school students in ideas;
3. complementation of acquiring manipulative skills and understanding basic concepts;
4. the encouragement and fostering of discovering generalizations through non-verbalized learning.³³

³¹Scott, Foreman and Co., op. cit., p. 30.

³²William T. Hale, "UICSM Decade of Experimentation", The Mathematics Teacher, LIV:615, Dec., 1961.

³³Scott, Foreman and Co., op. cit., p. 31.

School Mathematics Study Group

School Mathematics Study Group, or SMSG, represents the largest united effort of any study group for the improvement of the mathematics curriculum. This study group was organized in 1958 under the direction of Dr. Edward Begle of Yale University. This study group consists of a thirty-five member advisory committee, comprised of college and university mathematicians, high school mathematics teachers, experts in the field of education, and representatives of science and technology. The participants in various writing groups of SMSG include experienced high school teachers and distinguished research mathematicians. School Mathematics Study Group, which is presently located at Stanford University, is financed largely through National Science Foundation with an initial grant of one hundred-thousand dollars, and an additional grant amounting to over one-million dollars.

The textbooks of SMSG contain many new topics, in addition to many changes in the organization and presentation of older topics. Attention is focused on mathematical facts and skills, and the principles that provide a logical framework for them.³⁴ Some of the basic philosophical beliefs of School Mathematics Study Group are as follows:³⁵

³⁴ National Council of Teachers of Mathematics, op. cit., p. 31.

³⁵ National Council of Teachers of Mathematics, An Analysis of New Mathematics Programs, p. 33.

1. teaching of mathematics so that students will be able to learn new mathematical skills which the future will demand of them;
2. improvement of the mathematics curriculum so that it will not only offer students the basic mathematical skills, but also deeper understanding of the basic concepts and structures of mathematics;
3. more extensive and better training of teachers;
4. improvement of mathematics courses so that they are more attractive to students who are capable of studying mathematics.

Ball State Experimental Program

The Ball State Experimental Program was organized in 1955 under the direction of Dr. Merrill Shanks, then of Purdue University, Charles Brumfiel and Robert Eicholz, then of Ball State Teachers College. The program consists of curriculum revision for pupils, ranging from grades seven through twelve. The chief testing ground for this program is Burris (elementary and senior high school), which is the experimental school associated with Ball State.³⁶ The texts of the Ball State program are characterized by careful attention given to the logical development of the materials. The major purpose of the Ball State Experimental Program is to introduce the student to the axiomatic structure of mathematics.³⁷

³⁶Scott, Foresman, and Co., op. cit., p. 24.

³⁷National Council of Teachers of Mathematics, The Revolution in School Mathematics, p. 25.

All the current reform programs attempt to avoid the presentation of new materials as a string of unrelated topics. They stress many unifying themes or ideas in relation to structures, measurements, logical deductions, verbal generalization, graphical representation, and the development of the real numbers.³⁸ One of the basic ideas of the School Mathematics Study Group is that there exists a great deal of basic unity in mathematics and that this unity should be brought out. This group believes that mathematics should be "integrated." "A genuinely integrated treatment is one which not only brings out this unity, but also uses it to improve understanding."³⁹ According to the School Mathematics Study Group, various aspects of mathematics should be integrated, and not taught as separate entities. In algebra and geometry, many ideas can be integrated; Euclid failed to do this because algebra did not exist.⁴⁰ In the study of real numbers, for instance, one may think of the numbers as points on a line through a one-to-one correspondence, and thus real numbers become easier to visualize and understand. The geometry of lines is easier to understand if something is known about the real numbers. Such a connection is made immediately, hence students can make correlations from the beginning.

³⁸ Ibid., p. 22.

³⁹ University Symposium on Mathematics, Proceedings (1964), p. 438.

⁴⁰ Howard Ferr, "Reform of Mathematics Education Around the World," The Mathematics Teacher, LIV: 16, January 1965.

This view of School Mathematics Study Group is also a recommendation of the Commission on Mathematics of the College Entrance Examination Board. It is the Commission's view that the interrelation among various topics of mathematics should be pointed out and stressed. The University of Illinois Committee on School Mathematics places greater emphasis upon the consistency of the material rather than on the integration. The Ball State Experimental Program places greater emphasis upon the logical development.

In traditional geometry courses, plane and solid geometry are taught as separate entities. Solid geometry, in all of the contemporary programs, is no longer taught as a separate deductive theory course. Besides the shifting of emphasis in solid geometry, there also has been a change in the method of introducing space geometry. The consensus of the contemporary programs is that while the facts and the principles of space geometry are important, solid geometry is not a good place to study deductive proofs without the use of calculus. A full semester of space geometry is no longer considered a justifiable expenditure of the student's time. The important ideas of space geometry are approached earlier in the contemporary programs through plausible reasoning and intuition.⁴¹ The School Mathematics Study Group sequence of texts contains two strong units on space geometry in grade eight, and a unified, integrated treatment of plane and solid

⁴¹National Council of Teachers of Mathematics, op. cit., p. 63.

in grade ten. In addition, three-dimensional coordinates and vectors are introduced in grade eleven. According to School Mathematics Study Group, "A pupil who has completed this sequence has more functional knowledge than a pupil who has had the traditional course in deductive solid geometry, which contains no reference to vectors or three-dimensional coordinate systems."⁴² Exercises involving three-dimensional figures and space properties prevail throughout the plane geometry texts. Instead of a thorough deductive course, School Mathematics Study Group believes that students need to increase their ability to handle space configurations, to visualize figures in space, and to draw three-dimensional sketches. Though their programs are similar in many respects, School Mathematics Study Group believes the Commission's intuitive approach to solid geometry is not feasible.⁴³ School Mathematics Study Group gives the student a large amount of informal experience though they do not stress the deductive treatment.⁴⁴ In the geometry text of the Ball State Experimental Program, a summary of space geometry is presented in one chapter and an introduction to coordinate geometry in another, but this is out of the main line of development. Three-dimensional coordinate systems and vectors are not introduced until grade twelve in pre-cal-

⁴²Ibid., p.69.

⁴³Edwin Moise, op. cit., p. 442.

⁴⁴School Mathematics Study Group, Report of An Orientation Conference for SMSG Experimental Program, p. 18.

culus mathematics. The drawing of space configurations is not stressed, and plane geometry and solid geometry are not integrated.

In the University of Illinois program, there is not a great deal of interest in solid geometry and its integration with plane geometry. In regard to vectors, the University of Illinois Committee on School Mathematics, according to Dr. Beberman, expects to incorporate vectors as an approach to mathematics in grades seven and eight in the near future. They are currently using a vector approach to geometry in their high school curriculum.⁴⁵

In line with the changing emphasis on solid geometry, many traditional topics, such as involved proofs, extensive computations with logarithms, the use of trigonometry in the solution of oblique triangles, and long lists of mechanical exercises, are being omitted from the current programs. These topics, among others, are considered to be no longer in the mainstream of mathematical thought; they represent emphasis that is no longer appropriate to the needs of our society. The time which had been devoted to their presentation can now be used for more important topics and concepts.

In the treatment of analytic geometry with synthetic geometry, the general consensus of the study groups is that

⁴⁵University Symposium on Mathematics, Proceedings (1962), p. 11.

these two subjects should not be integrated. Synthetic geometry, even though it makes a contribution to analytic geometry, is not reversible as a contributing agent. Tenth grade proofs are easier to do synthetically, and hence analytic geometry actually need not be introduced until the end of the course.⁴⁶ All plane geometry texts contain a chapter on analytic geometry at the end of the book; the Commission on Mathematics recommends that analytic geometry be introduced after the first sequence of theorems in a plane geometry course. The School Mathematics Study Group feels that the commission's introduction of analytic geometry comes much too late to be of any help in the problems of foundations for geometry.⁴⁷ The commission also advocates analytic proofs, as well as synthetic proofs, in the solution of problems. The Secondary School Curriculum Committee, under the direction of the National Council of Teachers of Mathematics, advocates that the study of coordinate geometry should permeate the entire secondary sequence as early as grade seven with the one-to-one correspondence between the real numbers and points on line.⁴⁸ Thus, there are distinct differences of opinion concerning the introduction of analytic geometry.

⁴⁶Edwin Moise, op. cit., p. 442.

⁴⁷Ibid.

⁴⁸Scott, Foreman and Co., op. cit., p. 18.

Logic, as well as proofs, plays a major role in contemporary mathematics. The School Mathematics Study Group assumes that the students are aware of what a mathematical proof is, and that the students have had some experience with proofs.⁴⁹ The proofs put forth by this study group are generally complete and rigorous; but when the proofs are too involved, the authors state this and explain it in the teacher's commentary. Pedantic correctness is not the kind of rigor the group seeks; it seeks rigor in disclosure.⁵⁰ "A soundly rigorous treatment is one which brings out the crucial ideas as explicitly as possible."⁵¹ There is no formal logic in this group's program, but the "if-then" statements are included. The University of Illinois Committee on School Mathematics texts also assume experience in proofs prior to the level at which the texts are used. These texts tend to be as rigorous as possible, pointing out structural loopholes - points which are true but cannot be proved at the time. Ball State's geometry program is extremely rigorous, giving careful attention to logical development; both the algebra and geometry programs contain carefully constructed chapters on elementary logic. These chapters appear early in the texts, and the ideas developed in them are utilized continuously.⁵² In fact, a significant feature of both the al-

⁴⁹National Council of Teachers of Mathematics, op. cit., p. 43.

⁵⁰School Mathematics Study Group, op. cit., p. 18.

⁵¹Ibid, p. 18.

⁵²National Council of Teachers of Mathematics, op. cit., p. 20.

gebra and geometry programs is a full unit of logic in each.⁵³

In regard to precision of language in their texts, the Ball State and UICSM texts are the best. The University of Illinois Committee on School Mathematics considers it very important that the language of the teacher and of the text be clear, concise, and unambiguous. "Precise communication is the characteristic of a good textbook and a good teacher; correct action is a characteristic of a good learner."⁵⁴ The courses of the Ball State program and UICSM are generally concerned with precision in the use of the language of mathematics.⁵⁵ Both these programs are formalistic in their philosophical approaches; consistency is also very important in both. Definitions are formulated in precise terminology, and a small intuitive discussion of new terms and definitions appears before new materials. The University of Illinois Committee on School Mathematics program places great emphasis upon the inductive and concrete approaches to mathematics; the School Mathematics Study Group has a more nearly axiomatic approach, as does the Ball State geometry program.⁵⁶ In the texts of the School Mathematics Study Group, intuitive insight is encouraged throughout. This clarity and precision is essentially lacking in the traditional courses. A

⁵³William T. Hale, op. cit., p. 615.

⁵⁴National Council of Teachers of Mathematics, An Analysis of the New Mathematics Programs, p. 60.

⁵⁵University Symposium on Mathematics (1962), op. cit., p. 46.

⁵⁶Ibid.

question which often arises in this respect is how can so many high school graduates be so well grounded in mathematics despite what they have been told. According to the University of Illinois Committee on School Mathematics, children organize their mathematical knowledge in ways which are meaningful to them.⁵⁷ The contemporary mathematics programs are consistent and highly sequential, and are taught on a maturity level rather than on a grade level. The Commission on Mathematics of the College Entrance Examination Board also feels that content in mathematics, or in any other subject, must be taught on the level of maturity of the student.

Another important area of consideration is the methodology employed in the textbooks. The University of Illinois committee stresses the discovery of generalizations by the student with the delay of verbalization, even though precision of language is advocated. The emphasis is placed on the student's doing mathematics rather than being told about it. In other words, more education is stressed with less emphasis on structure.⁵⁸ Exploration exercises appear frequently in the texts and thus guide the student in the discovery of generalizations; the discovery method, according to the University of Illinois Committee on School Mathematics, results in power in mathematical

⁵⁷UIMP staff, "University of Illinois Mathematics Program," School Review, LXV: p. 458.

⁵⁸William T. Hale, op. cit., p. 616.

thinking and a developed interest in mathematics.

A common source of misunderstanding about the University of Illinois Committee on School Mathematics program is the type of discovery technique used. UICSM uses the non-verbalization awareness method rather than the strictly inductive method. The induction methods lead to confusion of sentence formation, and the verbalization causes much frustration at the advent of the discovery method.⁵⁹ Induction methods formulate generalizations too soon; it becomes a guessing game, and the students lack the ability to express themselves properly.⁶⁰ UICSM advocates the freedom of the student to attack problems in whichever fashion he chooses. Throughout the School Mathematics Study Group texts - geometry in particular - all the students are participants. The authors lead them through the intuitive processes to establish a conjecture and then a formal proof; Ball State's methodology is similar to that of SMSG.

The Commission on Mathematics feels that geometry should provide for original and creative thinking by students. A large part of the course should be devoted to original exercises involving both the discovery of relationships and their proofs.

Another area of general concern in the contemporary programs is "concept versus skill, or manipulation." In the traditional programs the emphasis is placed upon skills and applications. UICSM feels that acquiring manipulation and understanding

⁵⁹University of Illinois Committee on School Mathematics, "Learning by Discovery," The Mathematics Teacher, LIV: p. 226.

⁶⁰Ibid.

basic concepts are complementary activities. Manipulative skills should be used primarily to cast light upon basic concepts.⁶¹ Although a great emphasis is placed upon the development of concepts, a list of supplementary exercises appear at the end of each unit of the UICSM texts as an aid in the development of skills. Applications, especially in geometry, are not used to motivate the study of material, but a few exercises of this type, although they are few in number, do appear in the texts after the theory has been developed. One of the major objectives of the SMSG program is the development of concepts. The texts of SMSG do contain considerably more algebra than the usual texts, but they are concept-centered rather than manipulative-centered. The Ball State program agrees with SMSG in this respect.

The lack of emphasis upon physical applications is one of the major criticisms of these contemporary programs. Dr. Kline of New York University has issued sharp criticisms of the contemporary programs and has made recommendations which allow for more applications in the contemporary programs.

"Today there is a sharp distinction between applied and pure mathematics in formalism, rigor and abstractions. The curriculum of the high school is fashioned by pure mathematicians, and hence have been poisoned by pure mathematics." --Kline⁶²

⁶¹UIMP staff, op. cit., p. 459.

⁶²University Symposium on Mathematics (1964), op. cit., p. 22.

Dr. Kline believes that mathematics, presented as a subject, has no motivation, no meaning, and no purpose because the very purpose of mathematics - its physical applications - has been forgotten. Concept-centered mathematicians, says Dr. Kline, are teaching students the language of science by wallowing in vocabulary as if the introduction of new words will solve problems.⁶³ Only a few students will become pure mathematicians, but the greater number will be scientists and engineers; hence Dr. Kline believes that mathematics should be presented in the curriculum in terms of physical ideas. Today the high school mathematics programs stress deductive structures and axiomatics. The substance of mathematics - its physical background and meaning - is completely eroded.⁶⁴ "The students get, instead, a mathematics which is uninspiring and pointlessly abstract, and mathematics which is isolated from all other bodies of knowledge."⁶⁵

The reaction of Dr. Kline to the School Mathematics Study Group is that the program bars thinking far more extensively than do traditional mathematics programs, although he agrees that traditional mathematics programs are not good. It is his belief that SMSG loads the student with a great deal of sophisticated language that he is not prepared to absorb, thus he memorizes it and recites it as closely to verbatim as possible.⁶⁶

⁶³Ibid., p. 23.

⁶⁴Ibid., p. 22.

⁶⁵Ibid., p. 22.

⁶⁶Ibid., p. 67.

The SMSG program does not promote thinking, but hinders it, as compared with even the most unsatisfactory programs. Dr. Kline is more favorable toward UICSM; he especially praised Professor Beberman for tentative preparations of a new program which is science oriented. Dr. Ross of Ohio University, on the other hand, contends that within pure mathematics, most of the dilemmas which scientists face can be illustrated.⁶⁷ In pure mathematics the pupils' power of observation, their feeling for adventure involving conjecture, their critical sense in testing conjectures, and their capacity to use the process of proof to provide a measure of security for those conjectures which happen to be true are developed.⁶⁸ According to Dr. Ross, applications should be the result of understanding, and not a preamble to it, as Dr. Kline advocates.

In reply to these criticisms, Dr. Begle of SMSG and Dr. Beberman of UICSM commented in support of the contemporary programs. One of the major criticisms of SMSG, according to Dr. Begle, is that it does not pay much attention to the use of mathematics in the sciences.⁶⁹ Most students know little about the sciences when they reach the tenth grade; and it is very difficult to demonstrate the use of mathematics because the students are not familiar with the area of application. Most teachers also do not know enough science to teach it; this is one of the weaknesses in teacher training. In SMSG, physical situations

⁶⁷Ibid., p. 46.

⁶⁸Ibid.

⁶⁹University Symposium on Mathematics (1962), Proceedings, p.7.

occasionally are used in describing the theorems.⁷⁰ Dr. Beberman of UICSM says that the contemporary programs actually contain as many problems of application as the conventional.⁷¹ UICSM, as pointed out previously, expects to begin experimenting with the use of applications both as motivational material and as an end product. The Ball State authors contend that applied mathematics is a result of pure mathematics, hence their emphasis is upon pure mathematics.

A major difference among UICSM, Ball State's program, and MSG lies in their systems of postulation of their geometries, which attempt to correct the defects in Euclidean geometry. The Commission on Mathematics recommends no specific system of postulation, but feels that mathematics should consist of a set of undefined terms on which all other concepts are defined and propositions proved. Furthermore, the number of undefined concepts and unproved propositions are held to a theoretical minimum; for all practical purposes, propositions so obvious that proving them seems meaningless should be assumed from the beginning.⁷² Of all the contemporary programs, MSG most closely associates itself with this idea.

Ball State and UICSM use a Hilbert-like system of postulation. In the Ball State geometry program, the distinction between mathematical and physical geometry is pointed out to the

⁷⁰National Council of Teachers of Mathematics, op. cit., p. 42.

⁷¹University Symposium on Mathematics, op. cit., p. 12.

⁷²Report of the Commission on Mathematics, op. cit., p. 24.

students; the emphasis, of course, is on mathematical geometry. For example, it is observed that points and lines exist only in the mind; chalk marks are physical objects.⁷³ The terms draw, rotate, place, and move refer to physical activities, not to mathematics. In traditional texts these terms are not defined, and hence the significance is not made clear by postulates; Euclid did not make this differentiation. The Ball State geometry program is often described as "Euclid made precise and rigorous according to the standards of modern mathematics."⁷⁴ Proofs in this geometry must be justified by appealing to postulates, and not by pointing at physical objects. UICSM, although it too is based on Hilbert's postulates, is not as rigorous as the Ball State geometry.

Hilbert formulated his postulates at the beginning of the twentieth century. Hilbert's plan was to stay as near to the form of Euclid as possible and to supply precise postulates to serve as the basis of correct proofs of all Euclid's propositions. The entities - point, line, and plane - as well as the relations between incidence and congruence, are taken as undefined, but limited by precisely stated postulates. The first postulates of Hilbert are concerned with the incidence of points, lines, points and planes, and planes and lines; and fills in the gap which is left to the imagination in the elements of Eu-

⁷³Brumfiel, Eicholz, and Shanks, "The Ball State Experimental Program," The Mathematics Teacher, LIII: p. 80, Dec. 1961.

⁷⁴Ibid., p. 79.

clid.⁷⁵ The next set of postulates are the axioms of order, which involve the concept of betweenness, or order of points on a line as related to the real numbers. These ideas are then related to points in a plane; this leads to the development of the notion of separation of a plane by a line - a notion which Euclid said nothing about. Hilbert used the following axiom of Pasch: "A line which passes through a point between the two vertices A and B of a triangle ABC, either passes through vertex C, or a point between A and C, or a point between B and C."⁷⁶ The axioms of congruence in Hilbert's system, which pertain to congruent lines and segments, display the equality relationships and the operation of addition associated with the real numbers. To complete the postulate system, a parallel and continuity postulate is required. Hilbert used the Playfair form of the parallel postulate and an axiom of continuity known as the Law of Archimedes. SMSG, as well as the Commission on Mathematics, feels that a geometry program based on Hilbert's "Foundations of Geometry" is much too difficult and sophisticated for a tenth grader. SMSG does not reject Hilbert completely, but adopts the views of Birkhoff; it accepts the idea that the point, line, and plane should be undefined and adopts Hilbert's incidence axioms. Many of Hilbert's theorems can be proved easily by the Birkhoff

⁷⁵School Mathematics Study Group, Studies in Mathematics, p.1.27.

⁷⁶Ibid., p. 1.27.

approach. In particular, the notions of "between" and "segment" are closely related to the corresponding ideas in arithmetic. Birkhoff builds upon the student's knowledge of arithmetic, elementary algebra, and his ability to use the scale and protractor.⁷⁷ He also assumes the student's knowledge of the real numbers and their properties.

There are many reasons why SMSG favors the postulate system of Birkhoff. SMSG feels that Euclidean geometry genuinely deserves the place it has in a high school curriculum; students need to learn this particular subject, for Euclidean geometry introduces the student to concepts in the sense they are understood by mathematicians. Instead of starting from the beginning and axiomatizing everything as Hilbert did, Birkhoff assumes the real numbers are known; his geometric postulates are based on this assumption. SMSG believes that the advantage of Birkhoff's program is that the proofs are easier, and hence it is possible to avoid the conflict between logical correctness and psychological intelligence. Birkhoff's postulates stress the relationship between algebra and geometry; their integration lays a firm foundation for the introduction of analytic geometry.⁷⁸ Birkhoff's postulates have had a tremendous influence upon the philosophical aspects of SMSG. Instead of the axiom of Pasch, Birkhoff has adopted the Separation Postulate (see tables). These postu-

⁷⁷Ibid., p. 1.28.

⁷⁸Edwin Moise, op. cit., p. 439.

lates permit SMSG to supply the missing steps of Euclid's proof of exterior angles.⁷⁹ The Separation Postulate (of the plane by a line) also aids in the clarification of the concept of angles related to two intersecting lines.⁸⁰ Birkhoff also follows Hilbert's Playfair postulate: through a point not on a given line, there is at most one line parallel to the given line. Birkhoff also assumes the congruence relations as does Hilbert.

The major difference between SMSG and UICSM is where the emphasis is placed; SMSG stresses the reorganization of subject matter, while UICSM places more emphasis upon teacher training. Dr. Beberman of UICSM has said, "We are not as much concerned with the reorganization of the subject matter as we are with improving the teaching of the subject itself."⁸¹ One aspect that distinguishes the University of Illinois program from all the others is that it cannot be taught except by teachers who believe in this particular type of teaching method - the discovery technique. Both UICSM and SMSG advocate inservice teacher training. The teachers of the Ball State program have no special training, but they meet periodically for the discussion of pedagogical problems.

All the contemporary programs are directed toward the level of the college-capable student; homogeneous grouping is recommended. The Commission on Mathematics recommends that

⁷⁹School Mathematics Study Group, op. cit., p. 1.29.

⁸⁰Ibid.

⁸¹University Symposium on Mathematics (1962), op. cit., p. 10.

mathematics be comprised of homogeneous groups with students of similar interests and ability.⁸² College-bound students should not be exposed to certain so-called practical courses, for they can acquire this knowledge independently.⁸³ SMSG is now in the process of considering a curriculum for students who are of average ability and are not planning to enter college. The classes in the Ball State program consist of students with above average ability; the weaker students are doing as well as their counterparts in the traditional courses.⁸⁴ The Secondary School Curriculum Committee has recommended that certain provisions should be made in the classroom for individual differences. The same mathematical structures and concepts should be taught in all classes, but the amount, complexity, depth, and manner of organization and presentation should be varied accordingly. A program of enrichment should exist for the gifted, and a program of minimum essentials should be given to those students of below-average intelligence.⁸⁵

The greatest problem in carrying out a reform program is not necessarily that of defining its goals, but that of convincing others of the validity of these goals. The improved programs have not been in operation long enough to permit a statistical evaluation; however, available data shows that students in the improved programs do just as well as those in the traditional

⁸²Report of the Commission on Mathematics, op. cit., p. 24.

⁸³Ibid.

⁸⁴Scott, Foresman, and Co., op. cit., p. 25.

⁸⁵Ibid., p. 19.

courses. Tests were conducted by the Educational Testing Center and the Minnesota Testing Laboratory on the evaluation of SMSG.⁸⁶ These tests showed that SMSG students do as well on recognized standardized tests as those in the traditional courses, and that students in the SMSG programs had more knowledge concerning the understanding of fundamental concepts.⁸⁷ The retention of knowledge of the SMSG students was appreciably significant. Pupils with a lower scholastic level who were involved in the SMSG programs seemed to gain considerably over those in the conventional courses.⁸⁸ Some of the advantages of SMSG are as follows: thinking is encouraged, the privilege of writing proofs in a natural form is encouraged, opportunity is given for mathematical growth, and a greater appreciation for the structures of mathematics is developed. SMSG is trying to move toward a curriculum which involves preserving everything that is good in the present curriculums while, at the same time, introducing new approaches to basic concepts, structures, and thinking; the research will continue indefinitely.

UICSM is the only program that has been in operation long enough to permit the obtaining of information on students' performances in college. Evidence shows that graduates of the UICSM programs have done exceptionally well in college science and mathematics. Their scores on the advanced mathematics

⁸⁶University Symposium on Mathematics (1962), op. cit., p. 54.

⁸⁷Ibid.

⁸⁸Ibid.

achievement tests of the college boards are higher than ever before.⁸⁹ Some students have been exempted from certain mathematics courses and, in some instances, are enrolled in honors programs.⁹⁰ Some of the advantages of the UICSM program along with its discovery method are that it: 1) enables children who have had difficulty in mathematics to gain confidence in themselves; 2) is very effective in the lower grades as a source of motivation; 3) needs no elaborate justification of why mathematics is important - experience is the best teacher; and 4) results in rules which children have discovered, leaving a more permanent impression in their minds.

Some of the tentative evaluations of the Ball State program display its feasibility. Teachers with a strong mathematics background are very enthusiastic about this program as are many capable students.⁹¹ Those students who are well motivated, who have good study habits, and are enthusiastic about the program do not find the central ideas inaccessible.⁹² Many students who have real mathematical ability, but did not show it in the traditional methods of evaluation because of boredom with routine and memorization, are doing exceptionally well in contemporary mathematics because of its excitement and high level of interest.

⁸⁹National Council of Teacher of Mathematics, The Revolution in School Mathematics, p. 36.

⁹⁰Ibid.

⁹¹Scott, Foresman and Co., op. cit., p. 25.

⁹²Ibid.

From such an evaluation and comparison as this, one cannot conclude that any one program is better than another. All programs have their individualistic approaches, and all strive toward the betterment of the mathematics curriculum; hence, it is the common objective of these programs to establish a revised mathematics instruction plan - with regard to new content, the reorganization of old material, pedagogical techniques, philosophical concerns, teacher training programs, and shifts in emphasis - which will enable science to meet the future needs of our society. However, the responsibility for the improvement of the mathematics curriculum must not rest solely upon such study groups. It is the public's responsibility to be well-informed on these programs and their objectives so that our future society can enjoy the progress made possible by the current revolution in mathematical instruction.

Postulates

- I. The postulates of Euclid (basis for traditional plane geometry)*

Let the following be postulated:

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

*The following is taken from Euclid's Elements by Sir Thomas Heath.

- II. The postulates of School Mathematics Study Group based on Birkoff's ruler and protractor axioms*

1. Incidence postulates
 - a. Every line contains at least two points.
 - b. Every plane contains at least three distinct non-collinear points.
 - c. Space contains at least four distinct non-coplanar points.
 - d. Given two distinct points, there exists one and only one line containing them.
 - e. Given three distinct non-collinear points, there exists one and only one plane containing them.
 - f. If two distinct points lie in a plane, the line containing these points lies in the plane.
 - g. If two distinct planes intersect, their intersection is a line.
2. The ruler postulates
 - a. To every pair of points A, B there corresponds a unique real number, designated by AB, and called the distance between A and B. If A

and B are different points, then AB is positive. We allow also the possibility that $A = B$; in this case $AB = 0$.

- b. The points of a line can be put in one-to-one correspondence with the real numbers in such a way that the distance between two points is the absolute value of the difference between the corresponding numbers.
- c. If A and B are distinct points on a line L, then a coordinate system can be chosen on L, such that the coordinate of A is zero and the coordinate of B is positive.

3. The separation postulates

- a. If L is a line and p a plane containing L, the points of p not in L consist of two non-empty sets, called half-planes, such that if two points X and Y are in the same half-plane, the segment XY does not intersect the line L; and if X and Y are in different half-planes, the segment XY does intersect the line L.
- b. If p is a plane, the points not in p consist of two non-empty sets, called half-spaces, such that if two points X and Y are in the same half-space, the segment XY does not intersect the plane p; and if X and Y are in different half-spaces, the segment XY does intersect the plane.

4. The protractor postulates

- a. To every angle ABC, there corresponds a unique real number between 0 and 180 called the measure of angle and designated by $m(\angle ABC)$.
- b. Let QX be a ray on the edge of half-plane h. For any real number r between 0 and 180, there is a point Y in h such that $m(\angle XQY) = r$.
- c. If D is a point in the interior of $\angle AQB$, then $m(\angle AQD) + m(\angle BQD) = m(\angle AQB)$.
- d. If QA and QB are opposite rays and QC another ray, then $m(\angle AQC) + m(\angle BQC) = 180$.

5. The congruence postulate

- a. If there is a one-to-one correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding parts of the other, then the correspondence is a congruence.

6. The parallel postulate

- a. Through a given external point, there is at most one line parallel to a given line.

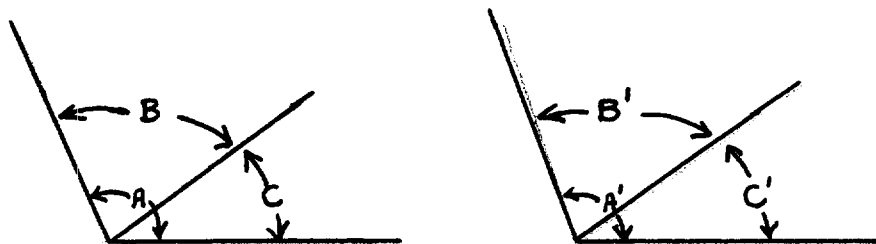
7. The area postulates
 - a. With every polygonal region, there is associated a unique positive real number, called the area of the region.
 - b. If two triangles are congruent, the triangular regions have the same area.
 - c. If a region R is the union of two non-overlapping regions S and T , then $\text{area } R = \text{area } S + \text{area } T$.
 - d. The area of a rectangular region is the product of the lengths of two adjacent sides.

*The SMSG postulates, as given in School Mathematics Study Group, Euclidean Geometry Based on The Ruler and Protractor Axioms.

III. The postulates of the Ball State Experimental Program based upon Hilbert's postulates.*

1. Incidence postulates
 - a. There are at least three points not all on a line.
 - b. For any two different points, there is exactly one line containing these points.
 - c. Every line contains at least two points.
2. Betweenness postulates
 - a. If B is between A and C , then A , B , and C are three different, or distinct, points on a line.
 - b. For every three points on a line, exactly one of them is between the other two.
 - c. Any four points on a line may be named A_1, A_2, A_3, A_4 , so that the only betweenness relations are the same as the order of the subscript numbers. That is, the betweenness relations are: A_2 between A_1 and A_3 ; A_2 between A_1 and A_4 ; etc.
 - d. If A and B are two points, then there is at least one point C such that B is between A and C , and at least one point D such that D is between A and B .
 - e. Every line separates the plane. By this we mean that all points of the plane not on the line are divided into two sets, called the two sides of the line, having the following properties:
 - (a) If P and Q belong to one of these sets, then no point of l is between P and Q . We say then that P and Q are on the same side of l .
 - (b) If P and Q are in different sets, then there is a point on l which is between P and Q .

3. Linear congruence postulates
 - a. Given points A and B on a line l and a point A' on a line l' , then on a given side of A' on l' there is exactly one point B' such that AB is congruent to $A'B'$. To indicate that AB is congruent to $A'B'$, we write " $AB \cong A'B'$." In congruence the order of the points does not matter; that is, if $AB \cong A'B'$, then also $AB \cong B'A'$, $BA \cong A'B'$, and $BA \cong B'A'$.
 - b. Congruence also satisfies the following:
 1. $AB \cong AB$.
 2. If $AB \cong A'B'$, then $A'B' \cong AB$.
 3. If $AB \cong A'B'$ and $A'B' \cong A''B''$, then $AB \cong A''B''$.
 - c. (This postulate tells how to "add" and "subtract" segments.) Suppose that B is between A and C on a line l , and that B' is between A' and C' on a line l' .
 1. If $AB \cong A'B'$ and $BC \cong B'C'$, then $AC \cong A'C'$.
 2. If $AC \cong A'C'$ and $BC \cong B'C'$, then $AB \cong A'B'$.
4. Archimedes' postulate
 - a. If AB is any segment on a line l and CD any other segment, then there is a finite number n of points A_1, A_2, \dots, A_n on l such that the points A, A_1, A_2, \dots, A_n are arranged in this order, and all the segments $AA_1, A_1A_2, \dots, A_{n-1}A_n$ are congruent to CD , and either $B = A_n$ or B lies between A_{n-1} and A_n .
5. Completeness postulate
 - a. For every line l and any given point A on l , and for any positive real number x , there is, on a given side of A , a point B such that $AB = x$.
6. Congruence postulates for angles
 - a. If $\angle A$ is not a straight angle and if r' is a ray from A' on a line l' , then on a given side of l' there is exactly one ray s' from A' such that the angle A' , with sides r' and s' , is congruent to angle A . We write this as " $\angle A' \cong \angle A$." If $\angle A$ is a straight angle, then the straight angle at A' with one side r' is the only angle with one side r' congruent to angle A .
 - b. $\angle A \cong \angle A$.
 If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.
 If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.
 - c. If in the figure any two of the pairs $\angle A$ and $\angle A'$, $\angle B$ and $\angle B'$, $\angle C$ and $\angle C'$ are congruent, then the third pairs of angles are congruent.
 In other words:
 1. If $\angle A \cong \angle A'$, and $\angle B \cong \angle B'$, then $\angle C \cong \angle C'$.
 2. If $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$, then $\angle A \cong \angle A'$.
 3. If $\angle C \cong \angle C'$, and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$.
 The angle A may be a straight angle.



- d. If for $\triangle ABC$ and $\triangle A'B'C'$; $AB \cong A'B'$, $AC \cong A'C'$, and $\angle A \cong \angle A'$, then $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$.
7. The parallel postulate
- a. Through a point not on a line there is no more than one line parallel to the given line.

*Brumfiel, Eicholz, and Shanks; Geometry.

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